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# On the flow past a sphere at low Reynolds number

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The flow of an incompressible, viscous fluid past a sphere is considered for small values of the Reynolds number. In particular the drag is found to be given by

$$D = D_S\{1+\tfrac{3}{8}R+\tfrac{9}{40}R^2(\log R+\gamma+\tfrac{5}{3}\log 2-\tfrac{323}{360})+\tfrac{27}{80}R^3\log R+O(R^3)\},$$

where $D_S$ is the Stokes drag, $R$ is the Reynolds number and $\gamma$ is Euler's constant.

## 1. Introduction

This is a classical problem with an extensive literature. To obtain higher order approximations beyond the first term given by Stokes (1851) is complicated by the fact that an expansion in terms of the Reynolds number, for the flow in the vicinity of the sphere, is not valid at large distances from the sphere. It has therefore to be matched with a separate expansion which is calculated for the 'outer' flow. The technique was evolved by Kaplun (1957). It has since been used by a number of investigators and is sufficiently well known not to require a separate account here. For a good historical survey of the problem of slow flow past a sphere, and a detailed description of the application of matched asymptotic expansions, the reader is referred to the paper by Proudman & Pearson (1957) who carried out the analysis as far as the term of order $R^2\log R$. The purpose of this paper is simply to continue the analysis of Proudman & Pearson as far as the term of order $R^3\log R$.

## 2. Basic equations

Let $a$ be the radius of the sphere, and let $U$ be the speed of the uniform streaming motion at infinity, assumed to be parallel to the positive $x$ axis of a system of co-ordinates based on the centre of the sphere. The velocity field, $U\mathbf{V}$, and the space co-ordinates can then be non-dimensionalized with the aid of $U$ and $a$ respectively, and the equations of motion will then be

$$\nabla^2\mathbf{V}-\nabla p = R(\mathbf{V}.\nabla)\,\mathbf{V}, \quad \nabla.\mathbf{V} = 0, \tag{2.1}$$

where $\rho\nu Up/a$ is the pressure, $R = Ua/\nu$ is the Reynolds number, $\rho$ is the density and $\nu$ is the kinematic viscosity. Alternatively one can express the governing equation in terms of a non-dimensional stream function $\psi$. This takes the form

$$\frac{1}{r^2}\frac{\partial(\psi, D^2\psi)}{\partial(r,\mu)}+\frac{2}{r^2}D^2\psi L\psi = \frac{1}{R^2}D^4\psi, \tag{2.2}$$

where $(r,\theta)$ are polar co-ordinates, $\mu = \cos\theta$ and

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}, \tag{2.3}$$

$$L = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \mu}. \tag{2.4}$$

## 3. The inner expansion

In terms of the stream function, the inner expansion is found to be of the form
$$\psi = \psi_0 + R\psi_1 + R^2 \log R\psi_{2L} + R^2\psi_2 + R^3 \log R\psi_{3L} + R^3\psi_3 + \ldots. \tag{3.1}$$
Given the form of the expansion, the various terms can be obtained by substitution of the series in (2.2) and integration of the resulting set of linear equations. One then finds that

$$\psi_0 = -\frac{1}{2}\left(2r^2 - 3r + \frac{1}{r}\right)Q_1(\mu), \tag{3.2}$$

$$\psi_1 = -\frac{3}{16}\left(2r^2 - 3r + \frac{1}{r}\right)Q_1(\mu) + \frac{3}{16}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)Q_2(\mu), \tag{3.3}$$

$$\psi_{2L} = -\frac{9}{80}\left(2r^2 - 3r + \frac{1}{r}\right)Q_1(\mu), \tag{3.4}$$

$$\begin{aligned}\psi_2 = &-\frac{3}{40}\left(c_1 r^2 + c_2 r + \frac{c_3}{r} - r^3 + 3r^2\log r + \frac{3}{16} - \frac{3\log r}{5r} - \frac{3}{16r^2} + \frac{1}{40r^3}\right)Q_1(\mu)\\ &+\frac{27}{32}\left(c_4 r^3 + c_5 + \frac{c_6}{r^2} + \frac{r^2}{3} - \frac{r}{2} - \frac{1}{6r}\right)Q_2(\mu)\\ &+\frac{9}{20}\left(\frac{c_7}{r} + \frac{c_8}{r^2} + \frac{r^3}{9} - \frac{43r^2}{120} + \frac{11r}{24} - \frac{1}{3} + \frac{4\log r}{35r} + \frac{1}{48r^2} + \frac{\log r}{42r^3}\right)Q_3(\mu),\end{aligned} \tag{3.5}$$

$$\psi_{3L} = d\left(2r^2 - 3r + \frac{1}{r}\right)Q_1(\mu) + \frac{27}{320}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)Q_2(\mu), \tag{3.6}$$

$$\begin{aligned}\psi_3 = \frac{27}{640}\Big\{&d_1 r^2 + d_2 r + \frac{d_3}{r} - (\tfrac{9}{2}c_4 - 2)r^3 - 6r^2\log r + (\tfrac{9}{2}c_5 - \tfrac{9}{8})\\ &+\frac{6\log r}{5r} - (\tfrac{1}{2}c_6 + \tfrac{1}{24})\frac{1}{r^2} - \frac{1}{20r^3}\Big\}Q_1(\mu) + \sum_{n=2}^{4} T_n(r)\,Q_n(\mu),\end{aligned} \tag{3.7}$$

where
$$Q_n(\mu) = \int_{-1}^{\mu} P_n(\mu)\,d\mu \tag{3.8}$$

and $P_n(\mu)$ is the Legendre polynomial of degree $n$. The precise form of the functions $T_n(r)$ in $\psi_3$ is not required.† For reference purposes we note that

$$Q_1 = (\mu^2-1)/2, \quad Q_2 = \mu(\mu^2-1)/2, \quad Q_3 = (\mu^2-1)(5\mu^2-1)/8. \tag{3.9}$$

The integration of the various equations for the $\psi_n$'s involves arbitrary constants in the complementary function, which are to be determined by the inner boundary conditions
$$\psi = \partial\psi/\partial r = 0 \quad \text{on} \quad r = 1, \tag{3.10}$$

† The expression for $\psi_2$ quoted by Proudman & Pearson (1957) is not correct.

and by appropriate matching with the outer solution. These constants are left undetermined in $\psi_2$, $\psi_{3L}$ and $\psi_3$. The calculation of $\psi_0$, $\psi_1$ and $\psi_{2L}$ has, however, already been discussed by Proudman & Pearson (1957) and these are quoted here in their final form with all the constants determined. These final forms have also been used to obtain the above expressions for $\psi_2$, $\psi_{3L}$ and $\psi_3$.

Briefly $\psi_0$ is the Stokes solution and is completely determined by the boundary conditions at $r = 1$ and the fact that it must match with a uniform stream at infinity. The next term in the outer solution is then obtained by matching with $\psi_0$. This in turn serves to determine $\psi_1$ and gives (3.3). To determine $\psi_{2L}$, Proudman & Pearson argue that a term such as $R^2r^2\log rQ_1$, which appears in $\psi_2$, can arise in the outer solution only from the combination $R^2r^2\log(Rr)\,Q_1$ (or $\rho^2\log\rho Q_1$ when expressed in terms of the outer variable $\rho = Rr$). Thus if a matching between the inner and outer solution is to be possible, $\psi_{2L}$ is required in order to combine suitably with the term $-(9r^2\log rQ_1)/40$ of $\psi_2$. In the next section the actual expression for the outer solution is given and the matching with $\psi_{2L}$ can then be checked directly.

## 4. The outer solution

The expansion for the velocity field in the outer solution is assumed to be of the form

$$\mathbf{V} = \mathbf{i}+\mathbf{V}_1+\mathbf{V}_2+\ldots, \tag{4.1}$$

where $\mathbf{i}$ is the unit vector parallel to the $x$ axis, and $\mathbf{V}_1$ satisfies the Oseen equations (Lamb 1932)

$$\nabla^2\mathbf{V}_1 - R\frac{\partial \mathbf{V}_1}{\partial x} - \nabla p_1 = 0, \quad \nabla.\mathbf{V}_1 = 0. \tag{4.2}$$

The solution has been discussed by Proudman & Pearson (1957) and will be quoted below.

The next term $\mathbf{V}_2$ satisfies the equations

$$\nabla^2\mathbf{V}_2 - R\frac{\partial \mathbf{V}_2}{\partial x} - \nabla p_2 = R(\mathbf{V}_1.\nabla)\,\mathbf{V}_1, \quad \nabla.\mathbf{V}_2 = 0. \tag{4.3}$$

If $\mathbf{V}_2$ is expressed in terms of a vector potential, such that

$$\mathbf{V}_2 = R\nabla\wedge\mathbf{A}_2, \tag{4.4}$$

then the equation to be satisfied by $\mathbf{A}_2$ is

$$\nabla^2\left(\nabla^2 - R\frac{\partial}{\partial x}\right)\mathbf{A}_2 = \nabla\wedge\{\mathbf{V}_1\wedge(\nabla\wedge\mathbf{V}_1)\} = \mathbf{F}_1 \ \ \text{(say)}, \tag{4.5}$$

and it may be verified that a particular solution is such that

$$\mathbf{A}_2 = \mathbf{A}_{21} - \mathbf{A}_{22}, \tag{4.6}$$

where
$$R\frac{\partial}{\partial x}\left(\nabla^2 - R\frac{\partial}{\partial x}\right)\mathbf{A}_{21} = \mathbf{F}_1, \quad R\frac{\partial}{\partial x}\nabla^2\mathbf{A}_{22} = \mathbf{F}_1. \tag{4.7}$$

Hence
$$\frac{\partial \mathbf{A}_2}{\partial x} = \frac{1}{4\pi R}\iiint \mathbf{F}_1(\mathbf{r}_1)\,\frac{1-\exp\left[\frac{1}{2}R(x-x_1-|\mathbf{r}-\mathbf{r}_1|)\right]d\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|}, \tag{4.8}$$

$$\mathbf{A}_2 = -\frac{1}{4\pi R}\iiint \mathbf{F}_1(\mathbf{r}_1)\int_0^{\frac{1}{2}R(|\mathbf{r}-\mathbf{r}_1|-x+x_1)}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha\,d\mathbf{r}_1. \tag{4.9}$$

Now it can easily be shown that
$$\mathbf{F}_1 = F_1\,\mathbf{i}_\phi, \tag{4.10}$$
where $\mathbf{i}_\phi$ is a unit vector in the direction defined by an angular increase about the $x$ axis. It follows that

$$\mathbf{A}_2 = A_2\mathbf{i}_\phi = -\mathbf{i}_\phi\frac{1}{4\pi R}\int_0^\infty r_1^2\,dr_1\int_0^\pi \sin\theta_1\,d\theta_1\int_0^{2\pi}\cos\phi_1\,d\phi_1$$
$$\times\left[F_1(r_1,\theta_1)\int_0^{\frac{1}{2}R\{(\mathbf{r}^2+\mathbf{r}_1^2-2trr_1)^{\frac{1}{2}}+r_1\cos\theta_1-r\cos\theta\}}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha\right], \tag{4.11}$$
where
$$t = \cos\theta\cos\theta_1+\sin\theta\sin\theta_1\cos\phi. \tag{4.12}$$

Finally, it is noted that the stream function associated with $\mathbf{A}_2$ is simply
$$Rr\sin\theta A_2.$$

The above equations contain all the information required to calculate the first three terms of the outer solution. We follow Proudman & Pearson and express the result in terms of the stream function and outer variable $\rho = Rr$. Then
$$R^2\psi = \Psi = \Psi_0(\rho,\mu)+R\Psi_1(\rho,\mu)+R^2\Psi_2(\rho,\mu)+\ldots, \tag{4.13}$$
where
$$\Psi_0 = \tfrac{1}{2}\rho^2(1-\mu^2) = \tfrac{1}{2}R^2r^2(1-\mu^2), \tag{4.14}$$
$$\Psi_1 = -\tfrac{3}{2}(1+\mu)\left(1-e^{-\frac{1}{2}\rho(1-\mu)}\right) = -\tfrac{3}{4}Rr(1-\mu^2)$$
$$+\tfrac{3}{16}R^2r^2(1-\mu^2)(1-\mu)-\tfrac{1}{32}R^3r^3(1-\mu^2)(1-\mu)^2+\ldots, \tag{4.15}$$
$$\Psi_2 = C(1+\mu)\left(1-e^{-\frac{1}{2}\rho(1-\mu)}\right)+\rho\sin\theta A_2$$
$$= \tfrac{1}{2}CRr(1-\mu^2)-\tfrac{1}{8}CR^2r^2(1-\mu^2)(1-\mu)+\tfrac{9}{32}Rr\mu(1-\mu^2)+\tfrac{1}{16}R^2r^2(1-\mu^2)$$
$$\times\left\{\tfrac{9}{5}\log(Rr)+\tfrac{9}{5}\gamma+3\log 2-\tfrac{747}{200}-\tfrac{9}{8}\mu+\tfrac{129}{400}(5\mu^2-1)\right\}+\ldots \tag{4.16}$$
and $\gamma$ is Euler's constant.

The first term, $\Psi_0$, is the stream function for a uniform stream and it is noted that it matches, as it should, with the leading term, for large $r$, of $\psi_0$. The second term in the outer solution, $\Psi_1$, is the stream function for a solution of the Oseen equations (4.2). In the strict application of the matching procedure, the constant of proportionality $(-\frac{3}{2})$ is chosen so that the leading term in the expansion of $\Psi_1$ for small $\rho$ matches with the second term of $\psi_0$, namely $3rQ_1(\mu)/2$. The next term $\Psi_2$ is a combination of the special solution obtained from $A_2$ and a complementary function. It turns out that the term displayed in (4.16) is a sufficient contribution from the complementary function, with a suitable value for $C$ obtained from matching. For this purpose the expansion of $A_2$ for small $\rho$ is required. Only the final result is quoted above. Some of the steps in the calculation are given in § 5. The final result has been checked independently by the two authors.

In the expansions of $\Psi_0$, $\Psi_1$, $\Psi_2$ in (4.14) (4.15), (4.16) all the terms which make a contribution to $\psi$ of order $R^2$ or larger are displayed. All these must be matched with corresponding terms in the inner solution. We note first that the expression for $\psi_{2L}$, quoted from Proudman & Pearson, does in fact check with the term of order $R^2 \log R$ which arises in the outer solution through $\Psi_2$. Next the constant $C$ of $\Psi_2$ is chosen so that the contribution $CRr(1-\mu^2)/2$ to $\psi$ from $\Psi_2$ matches with the term $9RrQ_1(\mu)/16$ of $R\psi_1$ in the inner solution. This gives

$$C = -\tfrac{9}{16}. \tag{4.17}$$

With this value of $C$, the rest of the terms in the outer solution can be matched with a suitable choice of the constants $c_1$ and $c_4$ of $\psi_2$ (equation (3.5)). Comparison of the appropriate terms in the inner and outer expansions shows that

$$3c_1/80 = \tfrac{1}{16}[\tfrac{9}{5}\gamma + 3\log 2 - \tfrac{747}{200} + \tfrac{9}{8}], \tag{4.18}$$

$$-27c_4/64 = -\tfrac{1}{16}. \tag{4.19}$$

The remaining constants in $\psi_2$, namely $c_2$, $c_3$, $c_5$, $c_6$, $c_7$, $c_8$, then follow from the boundary conditions $\psi = \partial\psi/\partial r = 0$ on $r = 1$. The final results are

$$c_1 = 3\gamma + 5\log 2 - \tfrac{87}{20}, \quad c_2 = -\tfrac{3}{2}\{3\gamma + 5\log 2 - \tfrac{191}{40}\}, \quad c_3 = \tfrac{1}{2}\{3\gamma + 5\log 2 - \tfrac{147}{40}\}, \tag{4.20}$$

$$c_4 = -\tfrac{4}{27}, \quad c_5 = \tfrac{29}{54}, \quad c_6 = -\tfrac{1}{18}, \tag{4.21}$$

$$c_7 = \tfrac{223}{3360}, \quad c_8 = \tfrac{353}{10080}. \tag{4.22}$$

This completes the inner solution as far as $\psi_2$. To proceed further we first consider those terms of the inner expansion which involve $\log R$ when expressed as a function of the outer variable. The significant terms are, from (3.4)–(3.7),

$$[-\tfrac{9}{80}R^2\log R(2r^2-3r) - \tfrac{9}{40}R^2r^2\log r + 2dR^3\log Rr^2 - \tfrac{81}{320}R^3r^2\log r]\,Q_1(\mu) + \tfrac{27}{320}R^3\log Rr^2Q_2(\mu), \tag{4.23}$$

$$= -\tfrac{9}{40}\rho^2\log\rho Q_1(\mu) + \mathrm{R}\log R(\tfrac{27}{80}\rho + 2d\rho^2 + \tfrac{81}{320}\rho^2)\,Q_1(\mu) + \tfrac{27}{320}R\log R\rho^2Q_2(\mu) + O(R), \tag{4.24}$$

when expressed as a function of the outer variable. The first term is already matched with a corresponding term in $\Psi_2$. The matching can be continued if the next term in the outer expansion is of the form $R^3\log R\Psi_{3L}$, where $\Psi_{3L}$ is derived from the Oseen equations. It is in fact sufficient to choose the expression

$$\begin{aligned}\Psi_{3L} &= N(1+\mu)\{1-e^{-\frac{1}{2}\rho(1-\mu)}\}\\ &= \tfrac{1}{2}N\rho(1-\mu^2) - \tfrac{1}{8}N\rho^2(1-\mu^2)(1-\mu) + \dots\\ &= -(N\rho - \tfrac{1}{4}N\rho^2)\,Q_1(\mu) - \tfrac{1}{4}N\rho^2Q_2(\mu) + \dots.\end{aligned} \tag{4.25}$$

The matching is then assured by the choice

$$N = -\tfrac{27}{80}, \quad d = -\tfrac{27}{160}, \tag{4.26}$$

which determines $\psi_{3L}$ and completes the analysis for both the inner and outer expansions.

## 5. The evaluation of $\Psi_2$

In equation (4.16) an expression for $\Psi_2$ to order $R^2$ was quoted. For completeness, some of the steps in the evaluation of this expression are now given.

That part of $\Psi_2$ which is derived directly from $A_2$ is, from (4.11),

$$-\frac{r\sin\theta}{4\pi}\int_0^\infty r_1^2\,dr_1\int_0^\pi \sin\theta_1\,d\theta_1\int_0^{2\pi}\cos\phi_1\,d\phi_1$$
$$\times\left[F_1(r_1,\theta_1)\int_0^{\frac{1}{2}R\{(r^2+r_1^2-2trr_1)^{\frac{1}{2}}+r_1\cos\theta_1-r\cos\theta\}}\frac{1-e^{-\alpha}}{\alpha}\,d\alpha\right], \quad (5.1)$$

where
$$t=\cos\theta\cos\theta_1+\sin\theta\sin\theta_1\cos\phi_1 \quad (5.2)$$

and

$$F_1(r,\theta)\,\mathbf{i}_\phi=\nabla\wedge\{\mathbf{V}_1\wedge(\nabla\wedge\mathbf{V}_1)\},$$

$$F_1(r,\theta)=-\tfrac{9}{8}\sin\theta\left[e^{-\frac{1}{2}Rr(1-\cos\theta)}\left\{\frac{R}{2r^3}(1-\cos\theta)+\frac{1}{r^4}(3-\cos\theta)+\frac{6}{Rr^5}\right\}\right.$$
$$\left.-e^{-Rr(1-\cos\theta)}\left\{\frac{R}{2r^3}(3+\cos\theta)+\frac{2}{r^4}(3+\cos\theta)+\frac{6}{Rr^5}\right\}\right], \quad (5.3)$$
$$=\tfrac{9}{8}\sin\theta\left[\frac{6\cos\theta}{r}-\frac{R}{4r^3}(5+6\cos\theta-19\cos^2\theta)+O(R^2)\right]. \quad (5.4)$$

To evaluate (5.1) the range of integration for $r_1$ is split into the two intervals $0\leqslant r_1\leqslant k$ and $k\leqslant r_1\leqslant\infty$, where $k$ is a constant such that $k\gg 1$, $Rk\ll 1$. The expression may then be written, with sufficient accuracy, in the form

$$\frac{Rr^2}{8\pi}\sin^2\theta\int_0^k r_1^3\,dr_1\int_0^\pi\sin^2\theta_1\,d\theta_1\int_0^{2\pi}\sin^2\phi_1\,d\phi_1$$
$$\times\left[\frac{F_1(r_1,\theta_1)}{(r^2+r_1^2-2trr_1)^{\frac{1}{2}}}\{1-\tfrac{1}{4}Rr[(r^2+r_1^2-2trr_1)^{\frac{1}{2}}+r_1\cos\theta_1-r\cos\theta]\}\right]$$
$$+\frac{r^2\sin^2\theta}{4\pi}\int_k^\infty r_1\,dr_1\int_0^\pi(1-\cos\theta_1)\,d\theta_1\int_0^{2\pi}\sin^2\phi_1\,d\phi_1$$
$$\times[F_1(r_1,\theta_1)\{1-e^{-\frac{1}{2}Rr_1(1+\cos\theta_1)}\}]. \quad (5.5)$$

To calculate the first term the approximate expression for $F_1$, given in (5.4), is used. The result is

$$\tfrac{9}{32}Rr\sin^2\theta\cos\theta-\tfrac{1}{16}R^2r^2\sin^2\theta\{\tfrac{9}{5}\log(k/r)+\tfrac{639}{400}+\tfrac{9}{8}\cos\theta-\tfrac{129}{80}\cos^2\theta\}+O(R^3). \quad (5.6)$$

Evaluation of the second term gives

$$\tfrac{1}{16}R^2r^2\sin^2\theta\{\tfrac{9}{5}\log(kR)+\tfrac{9}{5}\gamma+3\log 2-\tfrac{123}{50}\}+O(R^3)$$

and the combined contribution to $\Psi_2$ is therefore

$$\tfrac{9}{32}Rr\sin^2\theta\cos\theta+\tfrac{1}{16}R^2r^2\sin^2\theta$$
$$\times\{\tfrac{9}{5}\log(Rr)+\tfrac{9}{5}\gamma+3\log 2-\tfrac{1623}{400}-\tfrac{9}{8}\cos\theta+\tfrac{129}{80}\cos^2\theta\}+O(R^3). \quad (5.7)$$

## 6. The drag on the sphere

The result of primary interest is the drag on the sphere. This is evaluated as follows.

Let $\sigma_{rr}$ and $\sigma_{r\theta}$ be the non-dimensional tangential and normal stress components on the surface of the sphere, then the drag is given by

$$\begin{aligned} D &= 2\pi\rho\nu a U \int_0^\pi (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\, r^2 \sin\theta\, d\theta \\ &= \tfrac{1}{3}D_S \int_0^\pi \left\{\left(-p + \frac{2\partial V_r}{\partial r}\right)\cos\theta - \left(\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)\sin\theta\right\} r^2 \sin\theta\, d\theta, \end{aligned} \tag{6.1}$$

where the integral is to be evaluated at $r = 1$. This can be simplified, with the help of the boundary conditions to be satisfied at the surface of the sphere, to

$$D = \tfrac{1}{3}D_S \int_0^\pi \left\{-p\cos\theta + \frac{\partial^2\psi}{\partial r^2}\right\}_{r=1} \sin\theta\, d\theta. \tag{6.2}$$

The pressure can be determined, to within a constant which will not contribute to the drag, from the tangential component of the equation of motion

$$\begin{aligned} V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r}\frac{\partial^2}{\partial r^2}(rV_\theta) \\ &\quad + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V_\theta}{\partial\theta}\right) + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r^2\sin^2\theta}. \end{aligned} \tag{6.3}$$

On the surface of the sphere this gives

$$\frac{\partial p}{\partial \theta} = \frac{\partial^2}{\partial r^2}(rV_\theta), \tag{6.4}$$

or

$$p - p_0 = -\int^\theta \left(\frac{\partial^3\psi}{\partial r^3}\right)_{r=1} \frac{d\theta}{\sin\theta}. \tag{6.5}$$

Now the inner expansion for $\psi$ can be written in the form

$$\psi = \sum_{n=1}^{\infty} \Phi_n(r)\, Q_n(\mu). \tag{6.6}$$

It follows, from (6.5) and (6.2), that

$$p - p_0 = \sum_{n=1}^{\infty}\left(\frac{\partial^3\Phi_n}{\partial r^3}\right)_{r=1} \int^\mu \frac{Q_n(\mu)\, d\mu}{1-\mu^2} = -\sum_{n=1}^{\infty}\left(\frac{\partial^3\Phi_n}{\partial r^3}\right)_{r=1} \frac{P_n(\mu)}{n(n+1)}, \tag{6.7}$$

$$\begin{aligned} D &= \tfrac{1}{3}D_S \sum_{n=1}^{\infty} \int_{-1}^{1} \left\{\frac{\partial^3\Phi_n}{\partial r^3}\frac{\mu P_n(\mu)}{n(n+1)} + \frac{\partial^2\Phi_n}{\partial r^2} Q_n(\mu)\right\}_{r=1} d\mu \\ &= \tfrac{1}{9}D_S \left(\frac{\partial^3\Phi_1}{\partial r^3} - \frac{2\partial^2\Phi_1}{\partial r^2}\right)_{r=1} \end{aligned} \tag{6.8}$$

With the help of (6.8) and the inner expansion, the drag is easily calculated. The result is

$$D = D_S\{1 + \tfrac{3}{8}R + \tfrac{9}{40}R^2\{\log R + \gamma + \tfrac{5}{3}\log 2 - \tfrac{323}{360}\} + \tfrac{27}{80}R^3 \log R + O(R^3)\}, \tag{6.9}$$

where $D_S$ is the drag according to the Stokes solution, namely

$$D_S = 6\pi\rho\nu a U. \tag{6.10}$$

Figure 1 gives the theoretical results for $0 \leqslant R \leqslant 1$, together with the experimental measurements of Maxworthy (1965). The various curves show the effect

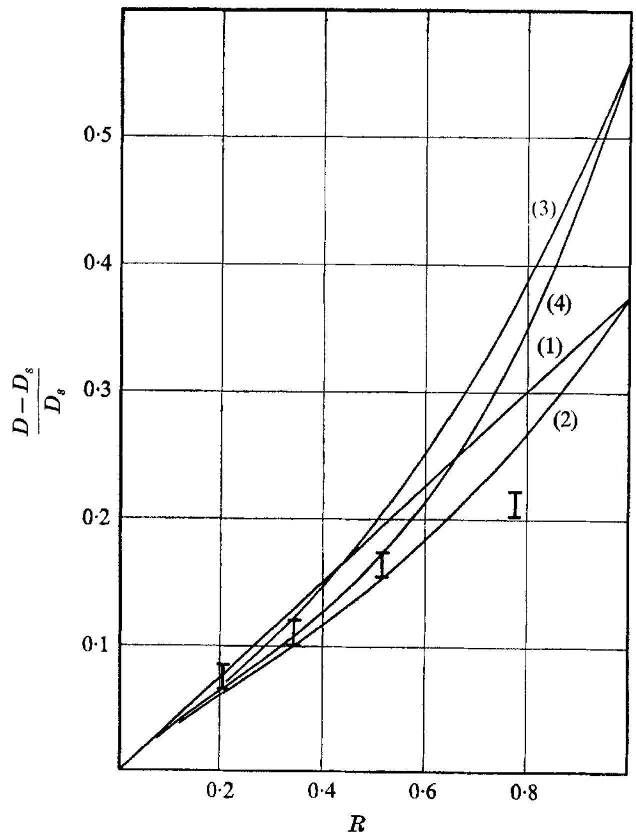


FIGURE 1. Experiment: I, Maxworthy (1965). Theory:

(1) $\dfrac{D-D_s}{D_s} = \dfrac{3R}{8}$; (2) $\dfrac{D-D_s}{D_s} = \dfrac{3R}{3} + \dfrac{9}{40} R^2 \log R$;

(3) $\dfrac{D-D_s}{D_s} = \dfrac{3R}{8} + \dfrac{9}{40} R^2 [\log R + \gamma + \tfrac{5}{3} \log 2 - \tfrac{323}{360}]$;

(4) $\dfrac{D-D_s}{D_s} = \dfrac{3R}{8} + \dfrac{9}{40} R^2 [\log R + \gamma + \tfrac{5}{3} \log 2 - \tfrac{323}{360}] + \tfrac{27}{80} R^3 \log R$.

of successive addition of a further term in the expansion. The conclusion seems to be that the expansion is of practical value only in the limited range

$$0 \leqslant R \leqslant 0{\cdot}5$$

and that in this range there is little point in continuing the expansion further.

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## Appendix. Modified computation of the drag coefficient of a sphere

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The calculation by Professors Chester and Breach of the term of order $R$ in the expansion of the drag coefficient $D$ for a sphere at small values of $R$ represents the first useful extension of the work of Oseen, since the earlier calculation of the term of order $R \log R$ by Proudman & Pearson (1957) was virtually useless without the accompanying term of order $R$. It is therefore particularly disappointing that the numerical 'convergence' of the expansion is so poor, and such as to limit its utility to the range $R < \frac{1}{2}$. The poor convergence is also rather surprising. One would not have expected any dynamical phenomena to develop in, say, the the range $1 < R < 10$, which were not approximately represented by the first few corrections to Stokes's solution for the flow; a view supported by observation at least at the lower end of the range, where measured values of the drag coefficient are in excess of Stokes's values by only 25 % or so.

It seems likely, therefore, that the poor convergence of the expansion (6.9) may, in part at least, be due to the unsuitability of the function $D$ for expansion in terms of $R$. The general nature of this function is known from observation over the whole range of Reynolds numbers for which the flow is laminar, and is such that $d(\log D)/d(\log R)$ increases monotonically from its value $-1$ at $R = 0$. Because of the onset of turbulence, the asymptotic value of this parameter, as $R \to \infty$, for steady flow is not known from observation; but it is presumably not positive, and, from arguments based on boundary-layer theory, not less than $-\frac{1}{2}$.

This behaviour suggests that a more appropriate form of presentation of results for $D$ might be

$$R = \epsilon (D/D_s)^m, \tag{1}$$

$$D_s/D = f_m(\epsilon), \tag{2}$$

where $m$ is a constant, and $\epsilon$ is a new expansion parameter defined by (1). From equation (6.9), the expansion of $f_m(\epsilon)$ for small values of $\epsilon$ is

$$f_m(\epsilon) \sim 1 - \tfrac{3}{8}\epsilon - \tfrac{9}{40}\epsilon^2 (\log \epsilon + \gamma + \tfrac{5}{3}\log 2 - \tfrac{548}{360} + \tfrac{5}{8}m) - \tfrac{27}{80}\epsilon^3 \log \epsilon + O(\epsilon^3). \tag{3}$$

If a large number of terms of (3) were available, it would be appropriate to attempt a prediction of $D$ for all Reynolds numbers by basing the choice of $m$ on the asymptotic behaviour of $D$ as $R \to \infty$. Thus, if $D \propto R^{-(m-1)/m}$ as $R \to \infty$, then $\epsilon \to$ constant $= \epsilon_m$ as $R \to \infty$, where $\epsilon_m$ is given by the first zero of $f_m(\epsilon)$. Thus, the expansion (3) would be relevant only in the finite range $(0, \epsilon_m)$, and, although one could not expect to determine the analytic behaviour of $f_m(\epsilon)$ in the neighbourhood of $\epsilon_m$ (corresponding to the asymptotic expansion of $D$ for large $R$), one might reasonably expect to obtain an estimate of the location of this zero (thus determining the coefficient of $R^{-(m-1)/m}$). In this context, the cases $m = 1$ and $m = 2$ are of special interest, since they correspond to the asymptotic behaviours $D \to$ constant and $D \propto R^{-\frac{1}{2}}$, respectively.

Unfortunately, the small number of known terms of (3) makes such an attempt over ambitious. In this case, the best results at small and moderate Reynolds numbers are to be expected from a choice of $m$ which corresponds to an asymptotic (for large $R$) behaviour somewhat closer to Stokes's law. This corresponds to $m > 2$.

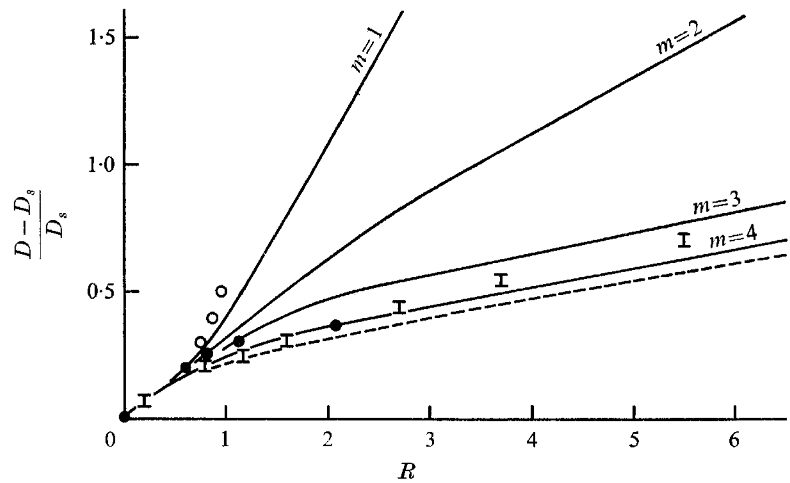


FIGURE 2. Modified computation of the drag coefficient. ———, based on all known terms of (3); - - - - -, based on first two terms of (3); ○, Chester & Breach; ●, points at which $\epsilon = \frac{1}{2}$; I, measurements by Maxworthy (1965).

The function $f_m(\epsilon)$ was computed from (3) for several values of $m$, and the corresponding results for the drag coefficient are shown in the accompanying figure. Some idea of the convergence of the expansion is given by the points (●) at which $\epsilon = \frac{1}{2}$, and by the broken curve, which, for $m = 4$, represents the effect of taking only the linear (Oseen) terms in (3). Agreement with the observations of Maxworthy (1965) is clearly best for a value of $m$ close to 4, and is then fairly good.

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